Ising machines as hardware solvers of combinatorial optimization problems

by **Naeimeh Mohseni, Peter L. McMahon, and Tim Byrnes**
(paper review)

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HP-Hard Problems

Non-digital approaches to solving them?

- Traveling salesman problem (TSP)
- Bin packing/Knapsack
- Boolean satisfiability (SAT)
- Graph coloring
- Subset sum
- Max cut
- …

HP-Hard Problems

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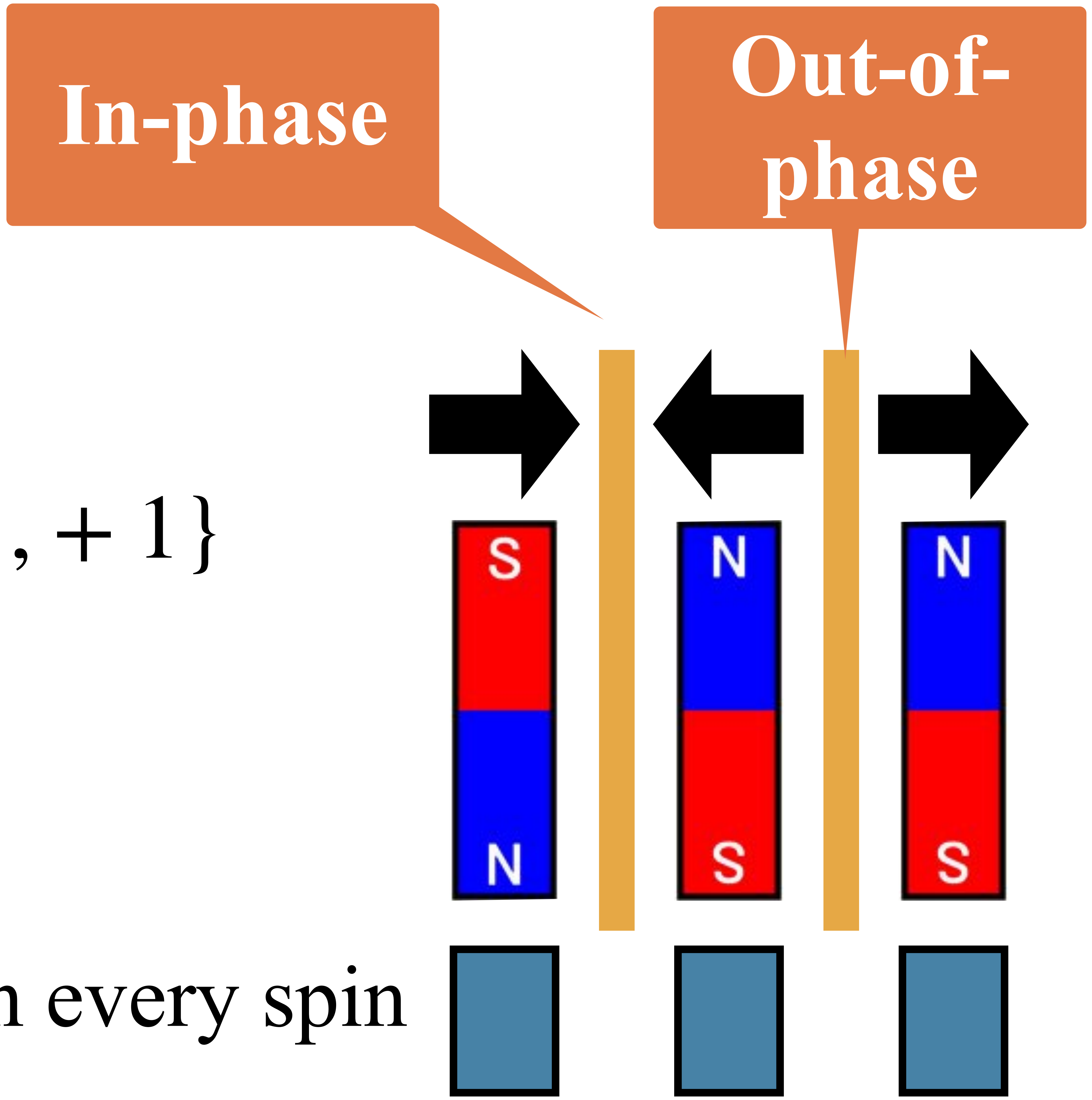
- Graph coloring
- Subs
- Max
- …

Can be mapped to ‘**Ising Hamiltonian**’ in polynomial time!

Ising Hamiltonian

$$H = -\sum_{i,j=1}^{N} J_{i,j} \cdot \delta_i \delta_j - \sum_{i=1}^{N} h_i \delta_i$$

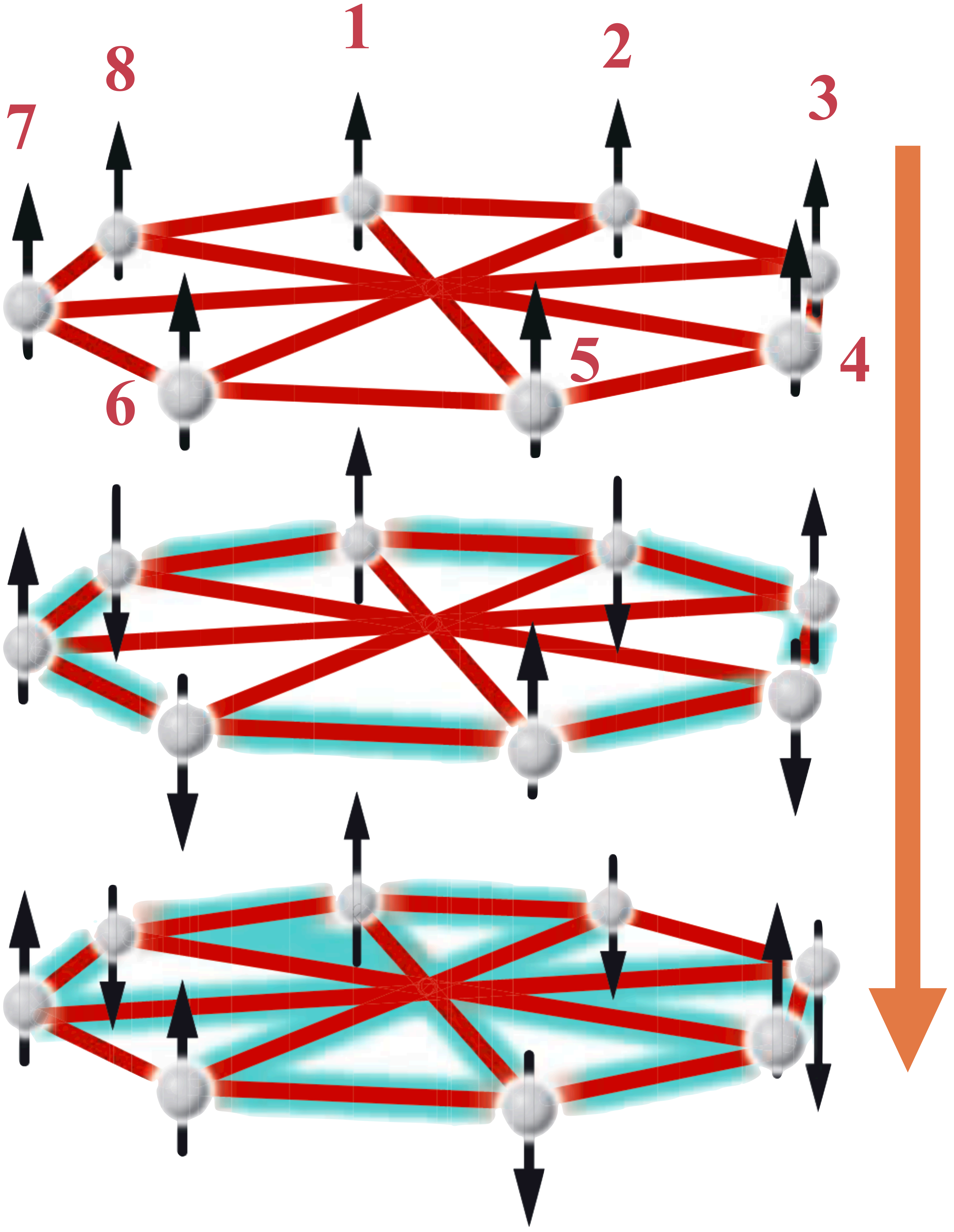
- “Spins”: $\vec{\delta} \triangleq (\delta_1, \delta_2, \ldots, \delta_N)$ where $\delta_k \in \{-1, +1\}$
  - **Interaction** $\delta_i \delta_j$: Bistable energy states
- **Connectivity/Coupling**: $J \in \mathbb{R}^{N \times N}$
- **Bias terms** $\vec{h}$: The external energy applied on every spin



Ising Hamiltonian

$$H = -\sum_{i,j=1}^{N} J_{i,j} \cdot \delta_i \delta_j$$

$$J = -\begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{array}$$



High energy state

Medium energy state

Low energy state

🟥 Spins want to be opposite 🟦 Satisfied interactions

[Cho, Science '16]

Ising Hamiltonian

$$H = -\sum_{i,j=1}^{N} J_{i,j} \cdot \delta_i \delta_j$$

$$J = -\begin{array}{c c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} & \\
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
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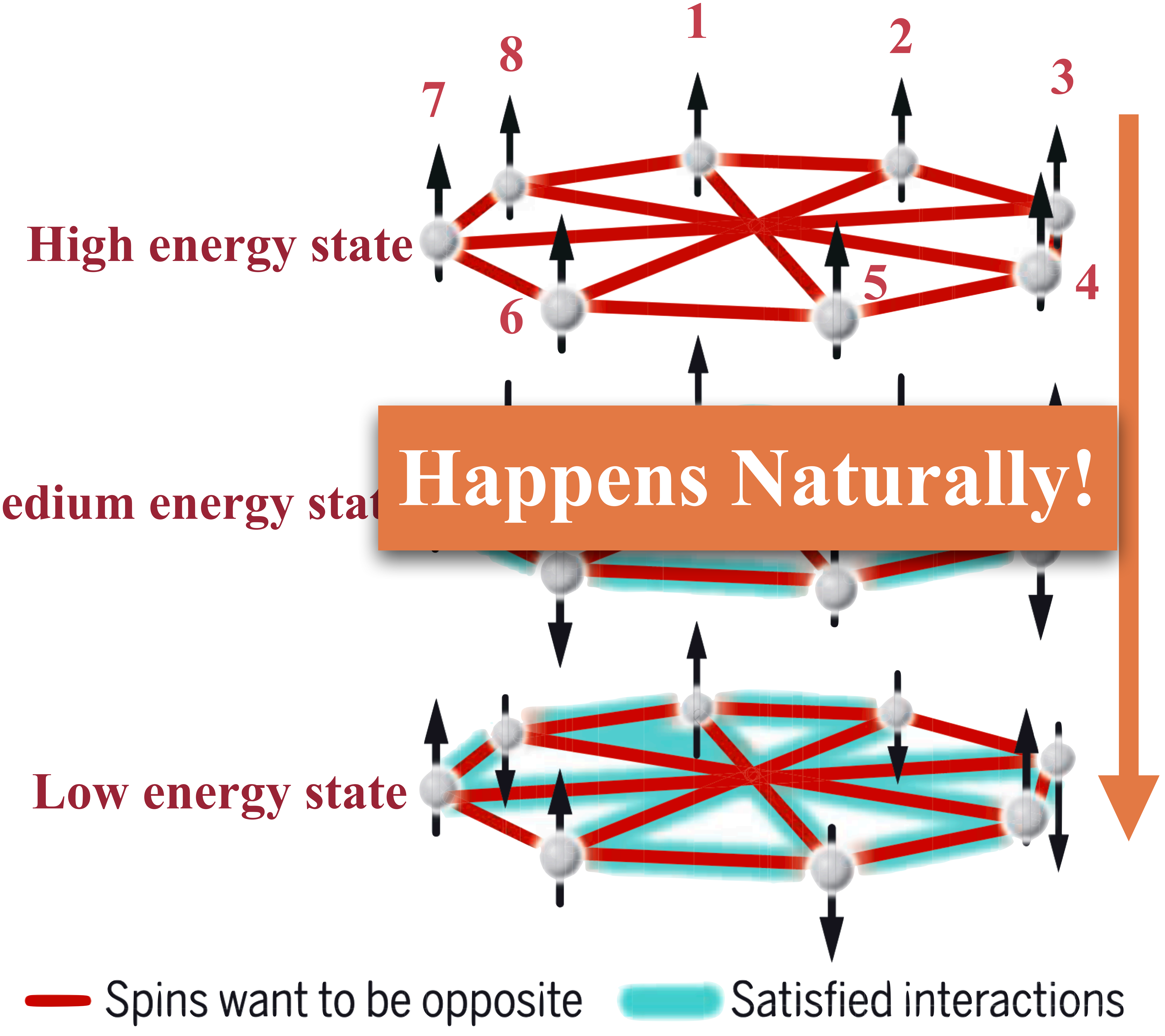
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\end{array}$$



High energy state

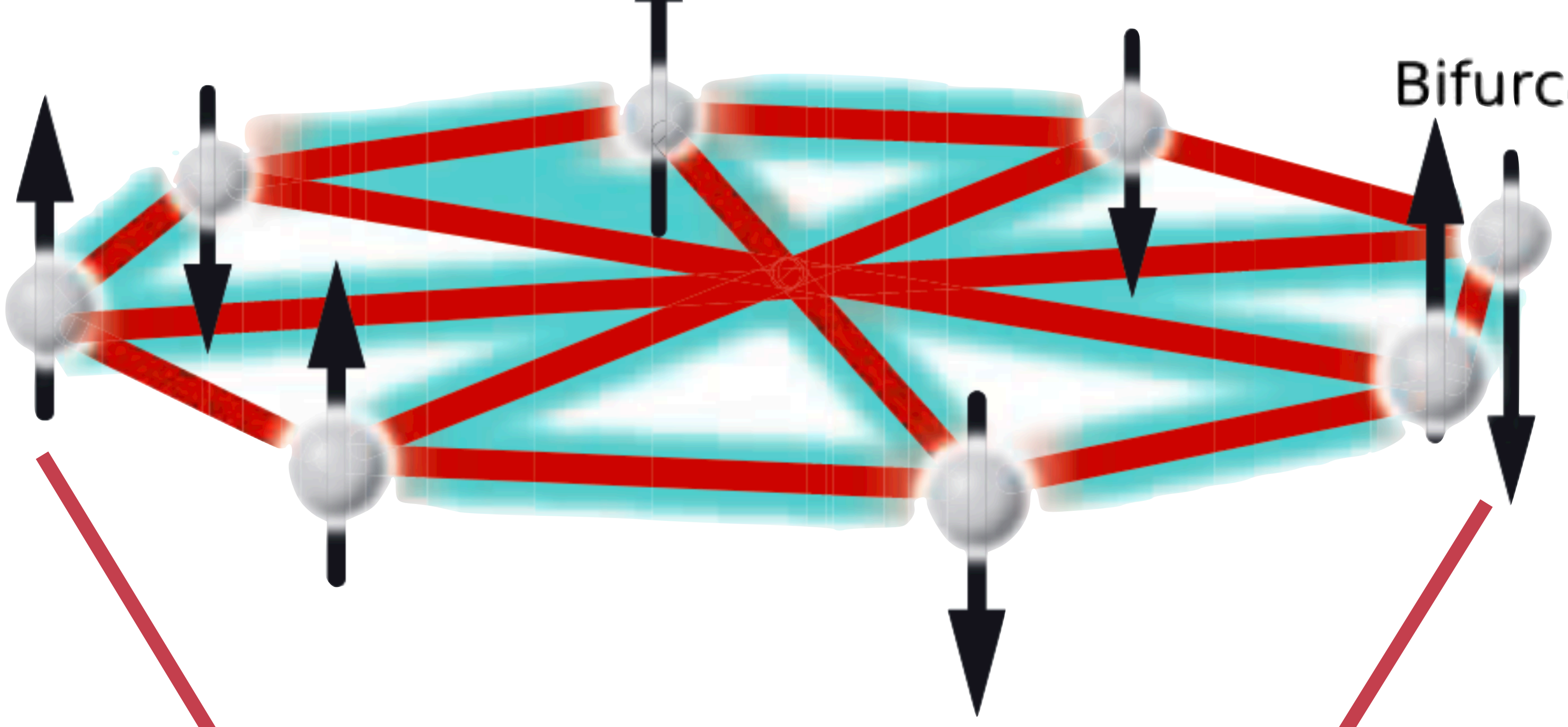
Medium energy stat

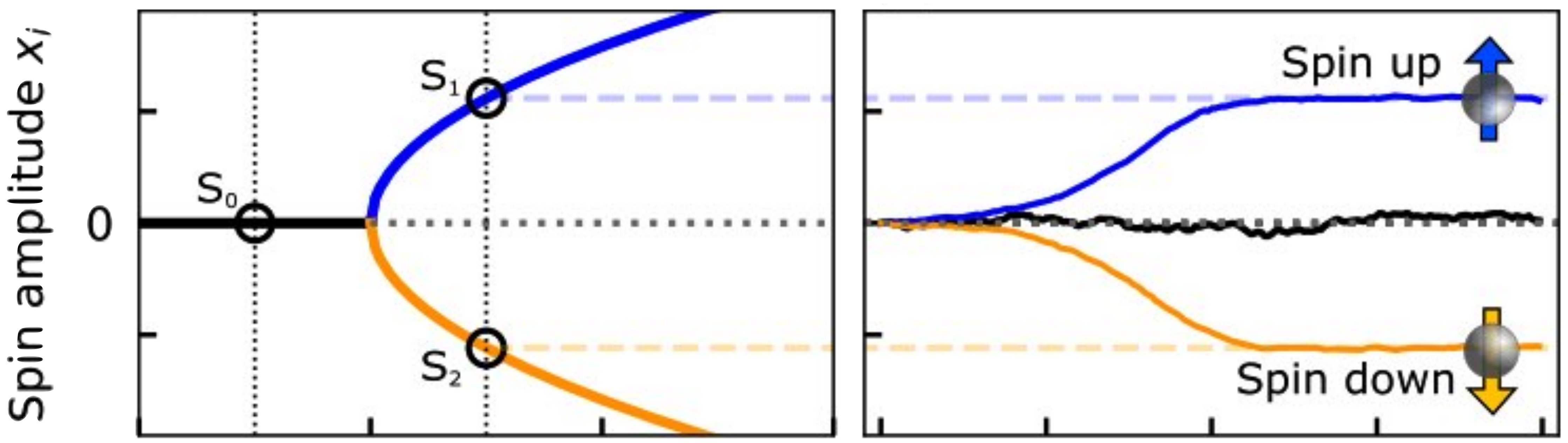
Happens Naturally!

Low energy state

[Cho, Science ’16]

Ising Machine





[Fabian Böhm et al., Communications Physics '21]

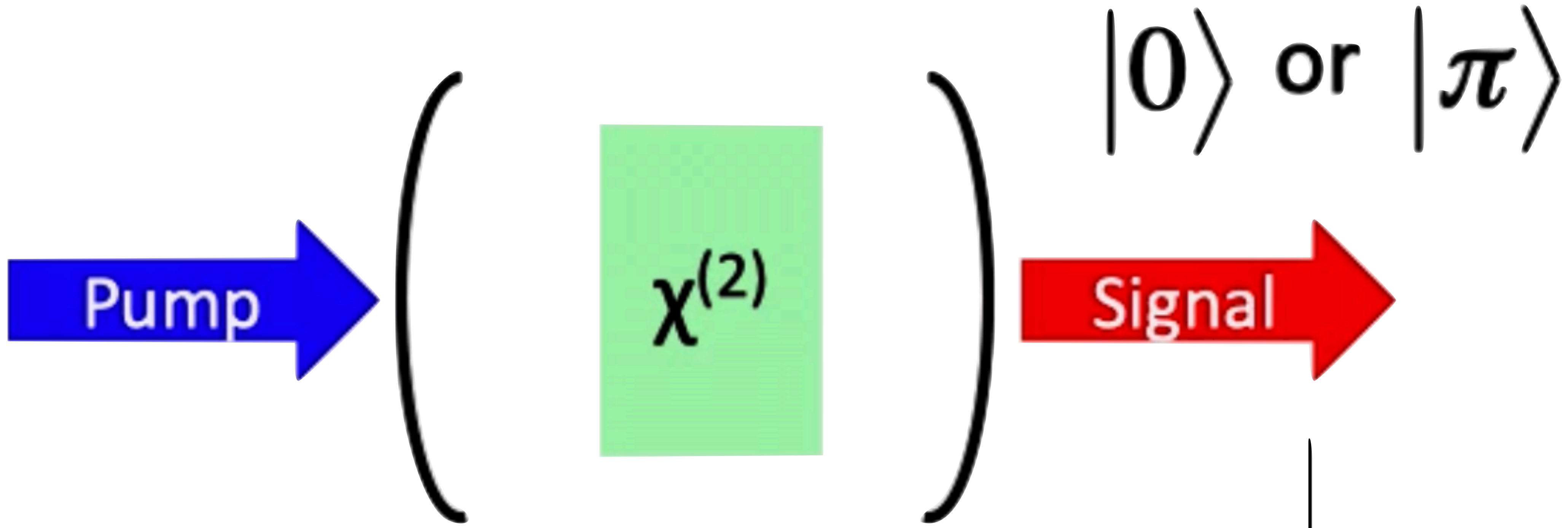
Ising Hamiltonian → **Physical System** → **Measurement**

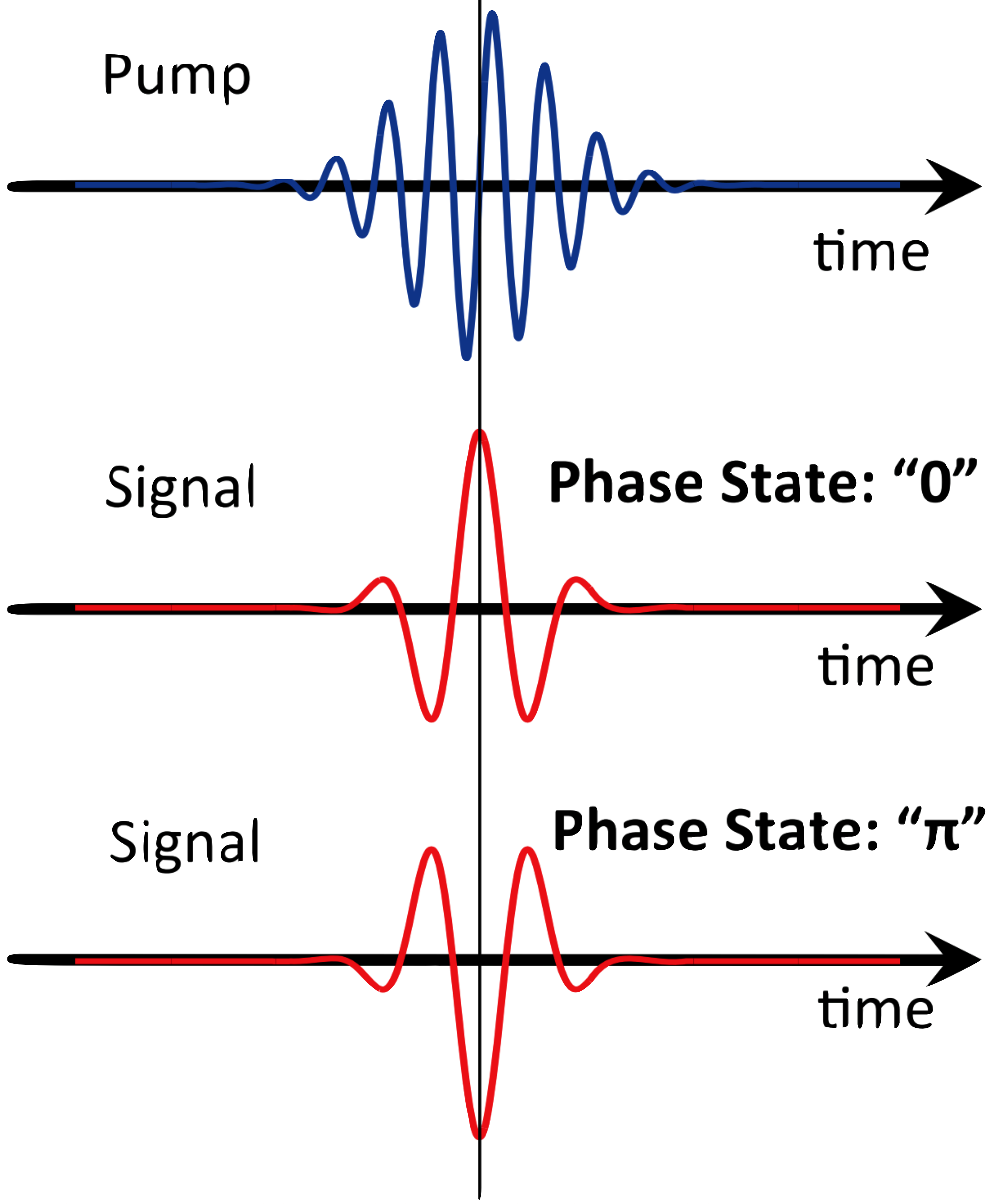
Energy → **Physical System**

Oscillator-Based Ising Machines

Degenerate Optical Parametric Oscillator (DOPO)

- $\vec{\delta}$: Phases of the oscillators
  - Resonant at ω_s
  - Bi-phase states: $|0\rangle$ or $|\pi\rangle$



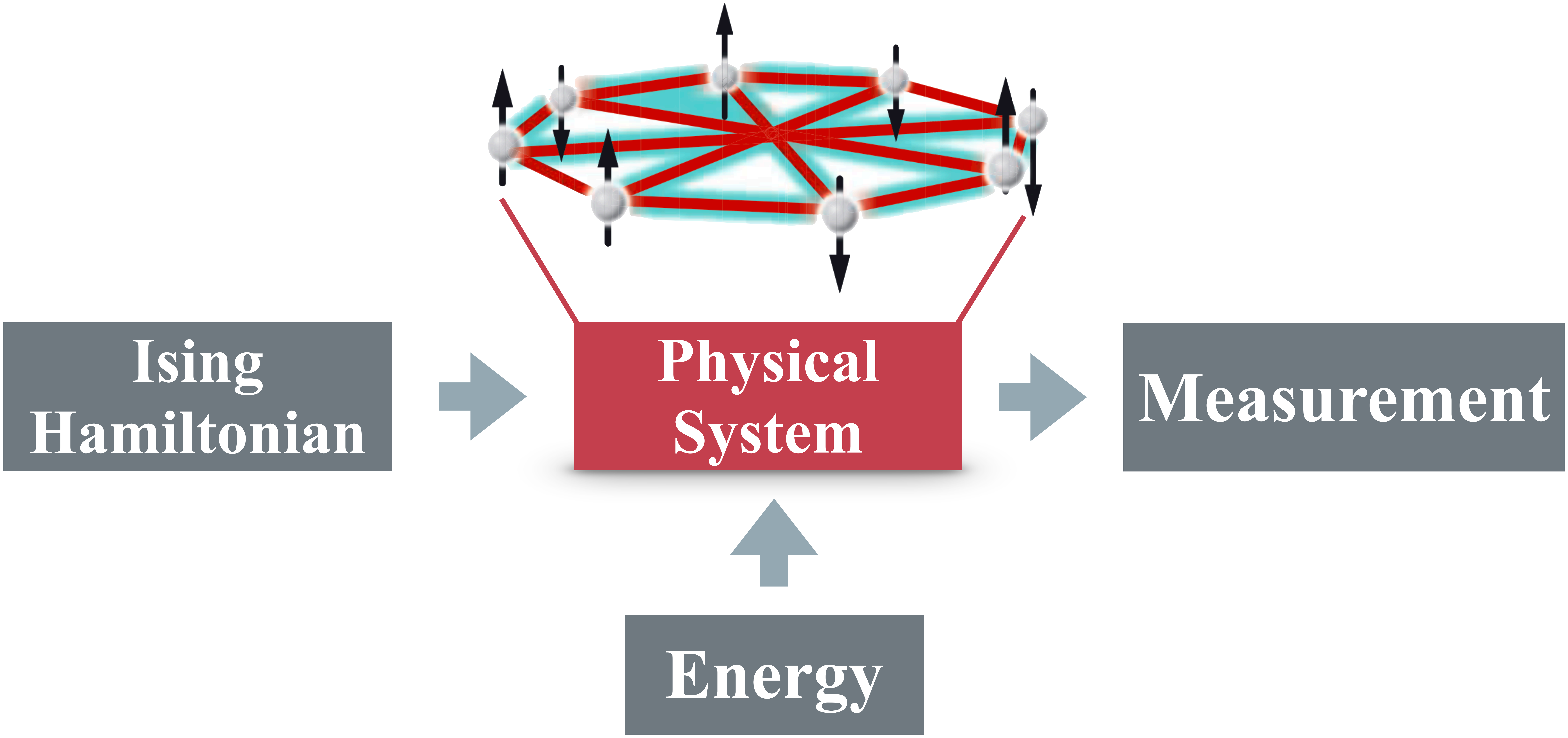


- Properties
  - Frequency locking: $\omega_p = 2 \cdot \omega_s$
  - Phase locking:

$$\phi_p = 2\boxed{(\phi_s + 0)} + \pi/2 + 2m\pi$$
$$\phi_p = 2\boxed{(\phi_s + \pi)} + \pi/2 + 2m\pi$$

[R Hamerly et al. Science Advances '19]

Coherent Ising Machine (CIM)



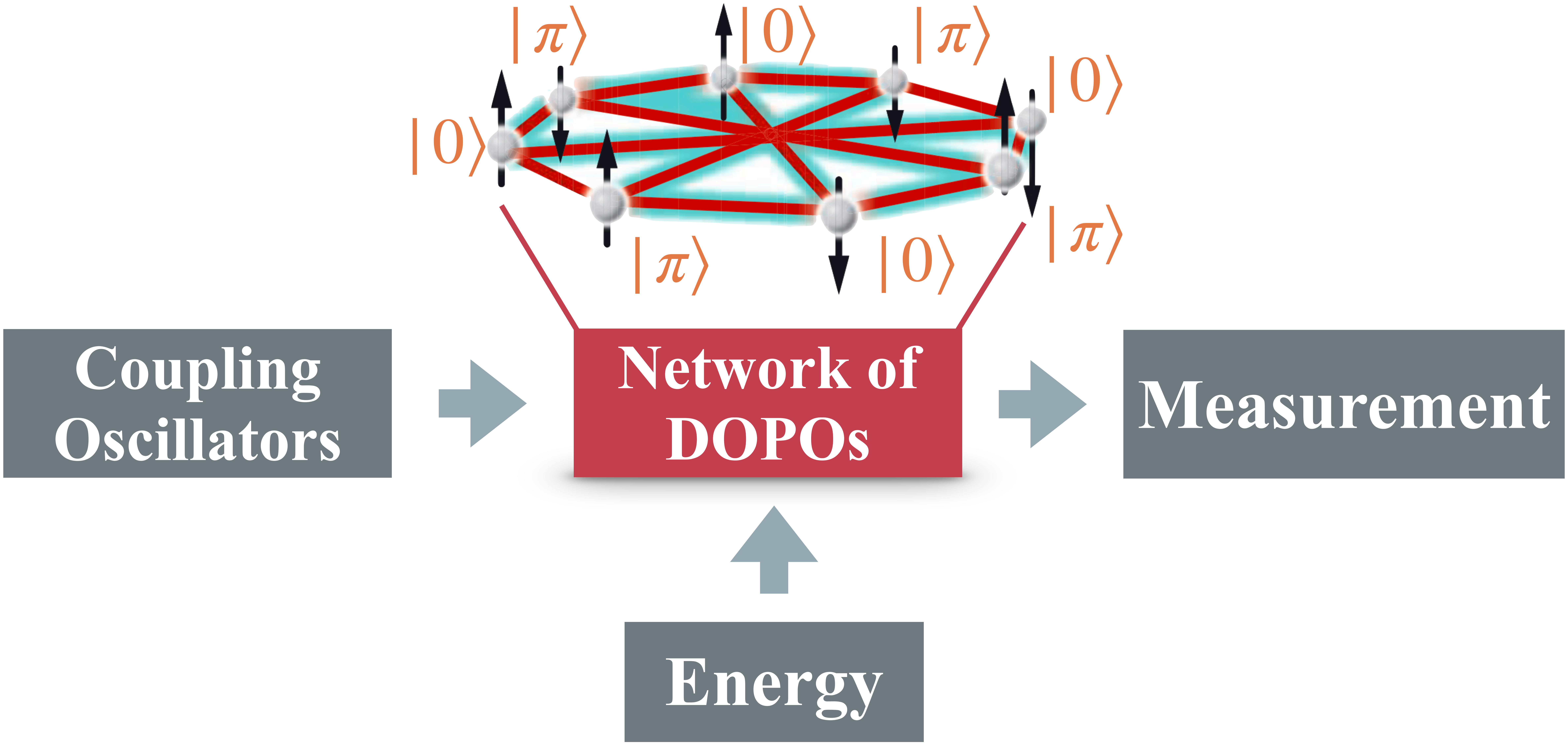
Ising Hamiltonian

Physical System

Measurement

Energy

Coherent Ising Machine (CIM)



Coupling Oscillators

Network of DOPOs

Measurement

Energy

HP-Hard Problems

- Traveling salesman problem (TSP)
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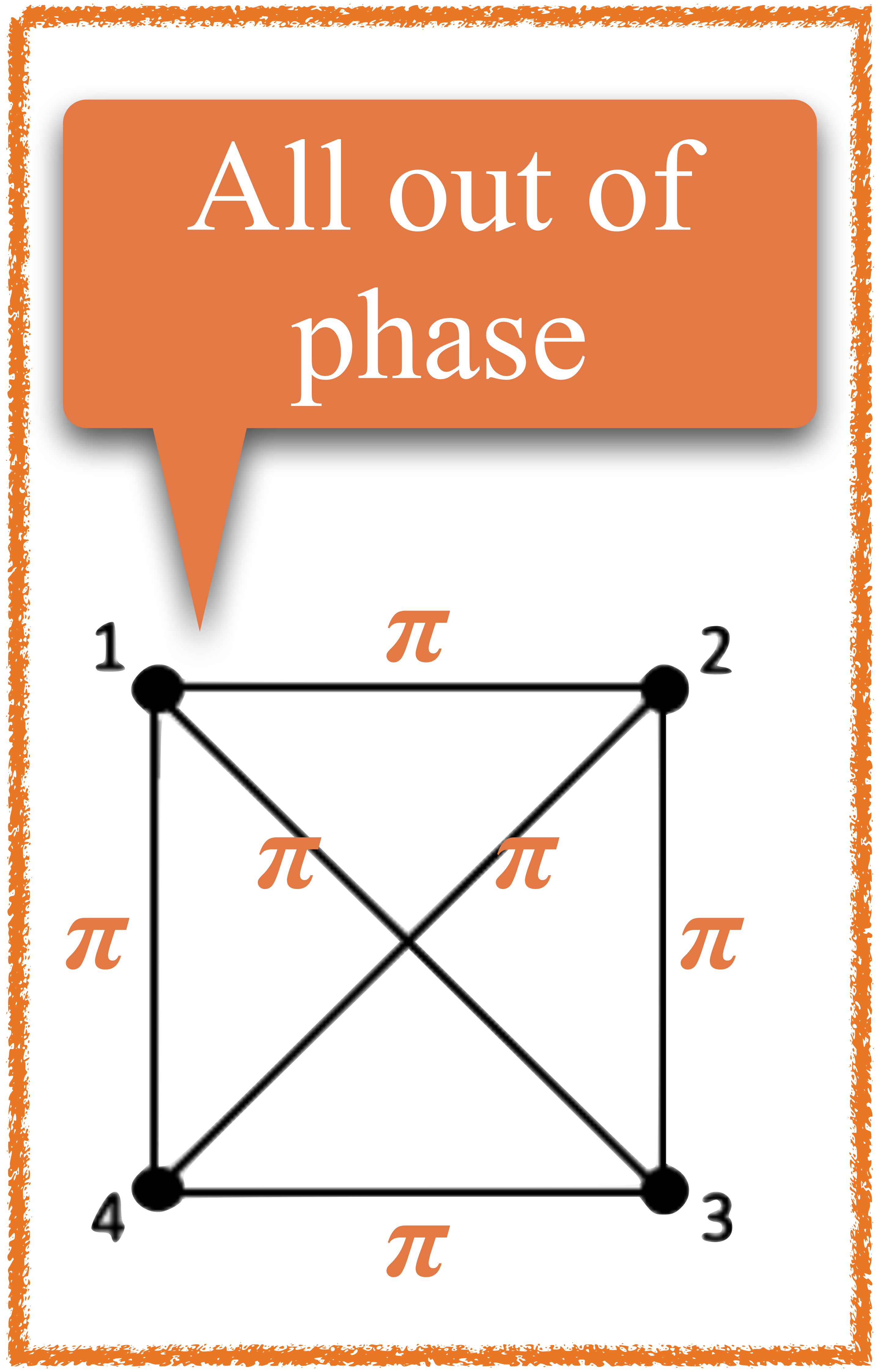
- …

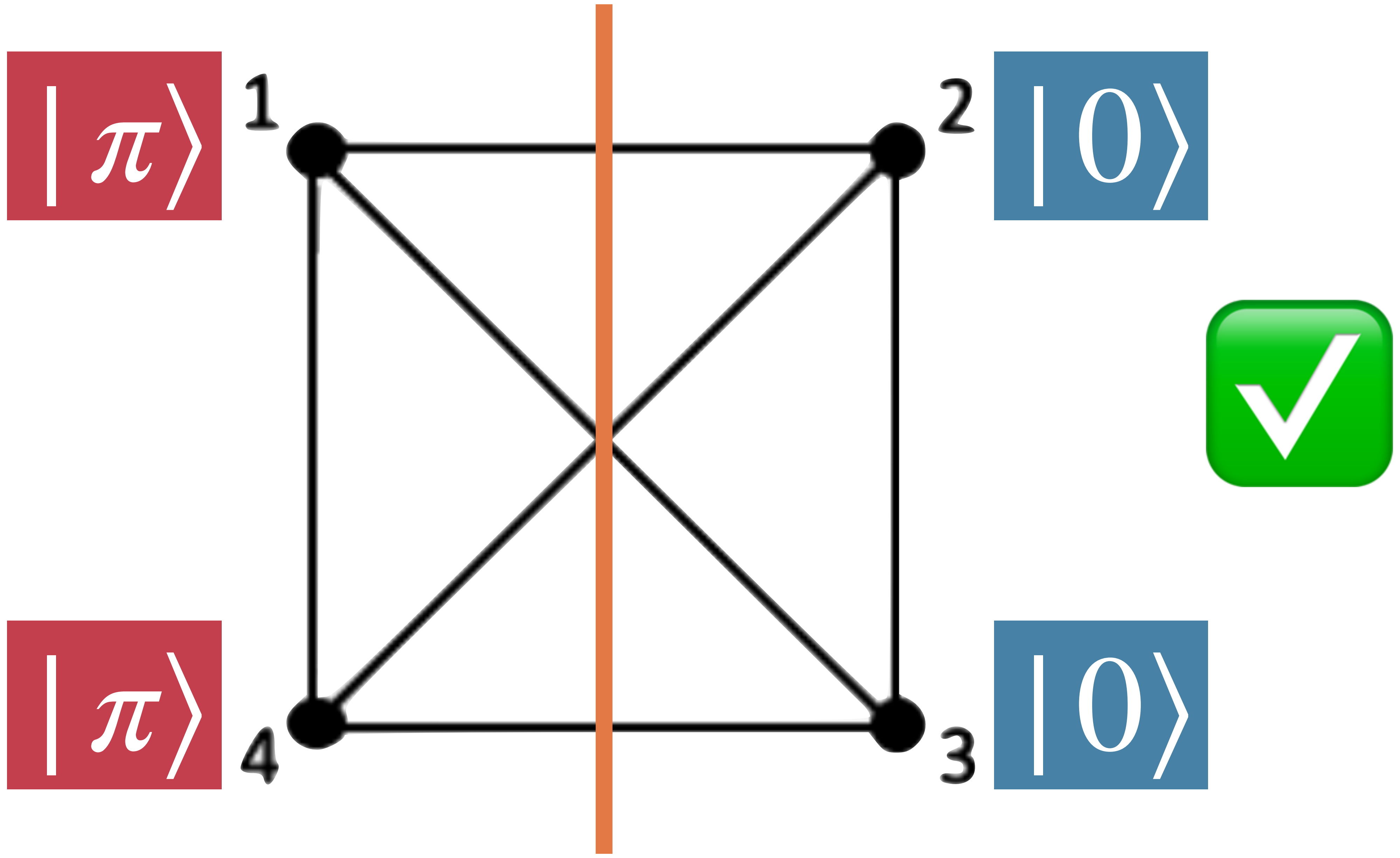
Mapping Max Cut to CIM

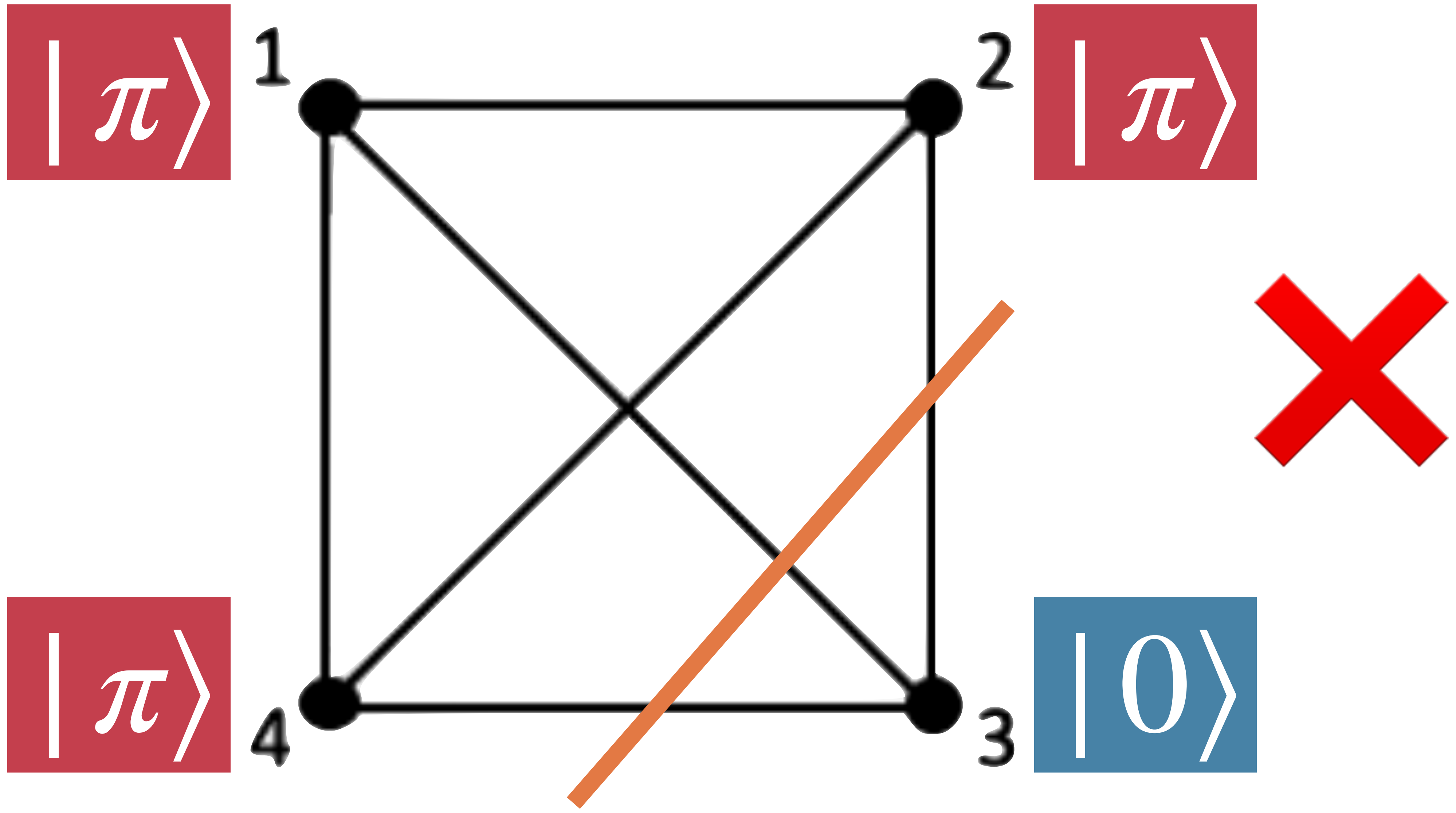
$$H = -\sum_{i,j=1}^{N} J_{i,j} \cdot \delta_i \delta_j$$

Initial coupling:

$$J = -\begin{bmatrix} 0 & \pi & \pi & \pi \\ 0 & 0 & \pi & \pi \\ 0 & 0 & 0 & \pi \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

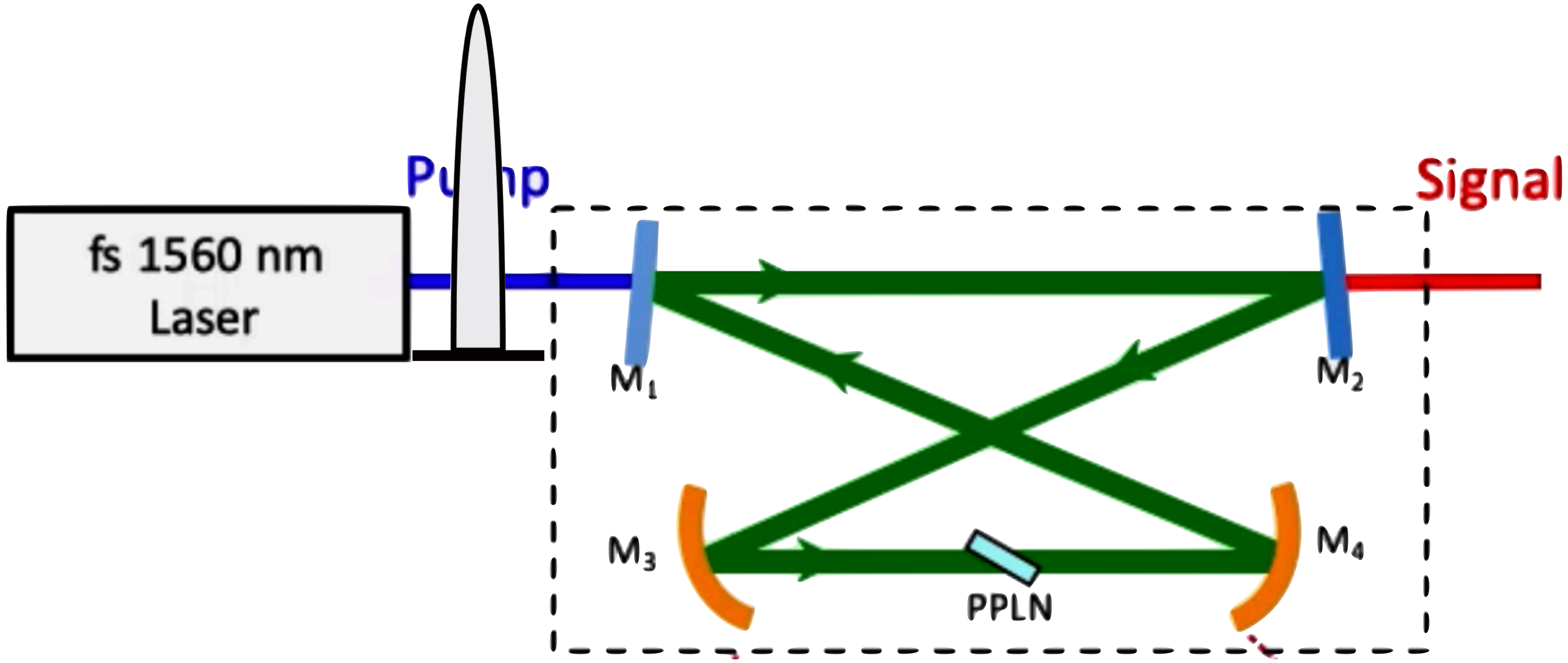

All out of phase





Programming the CIM

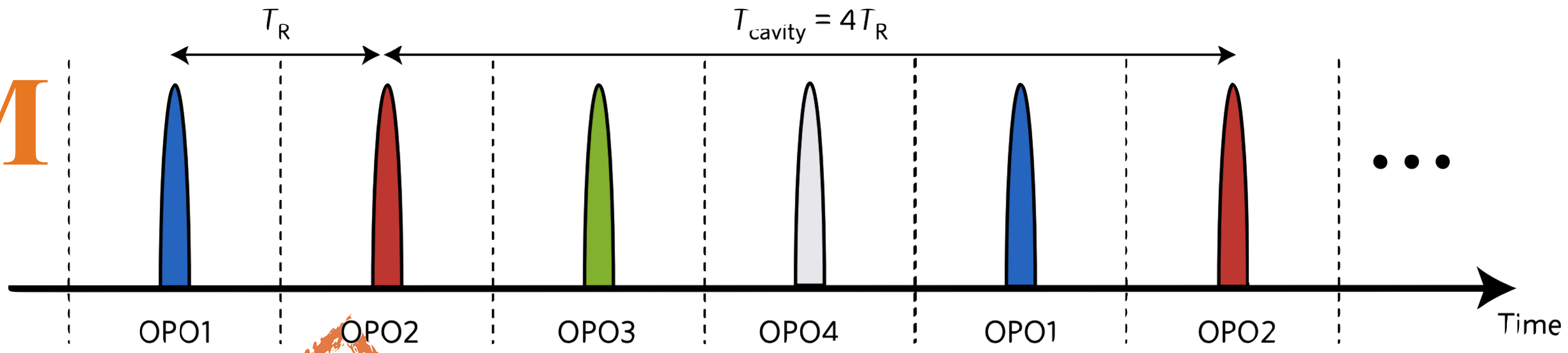
A single DOPO

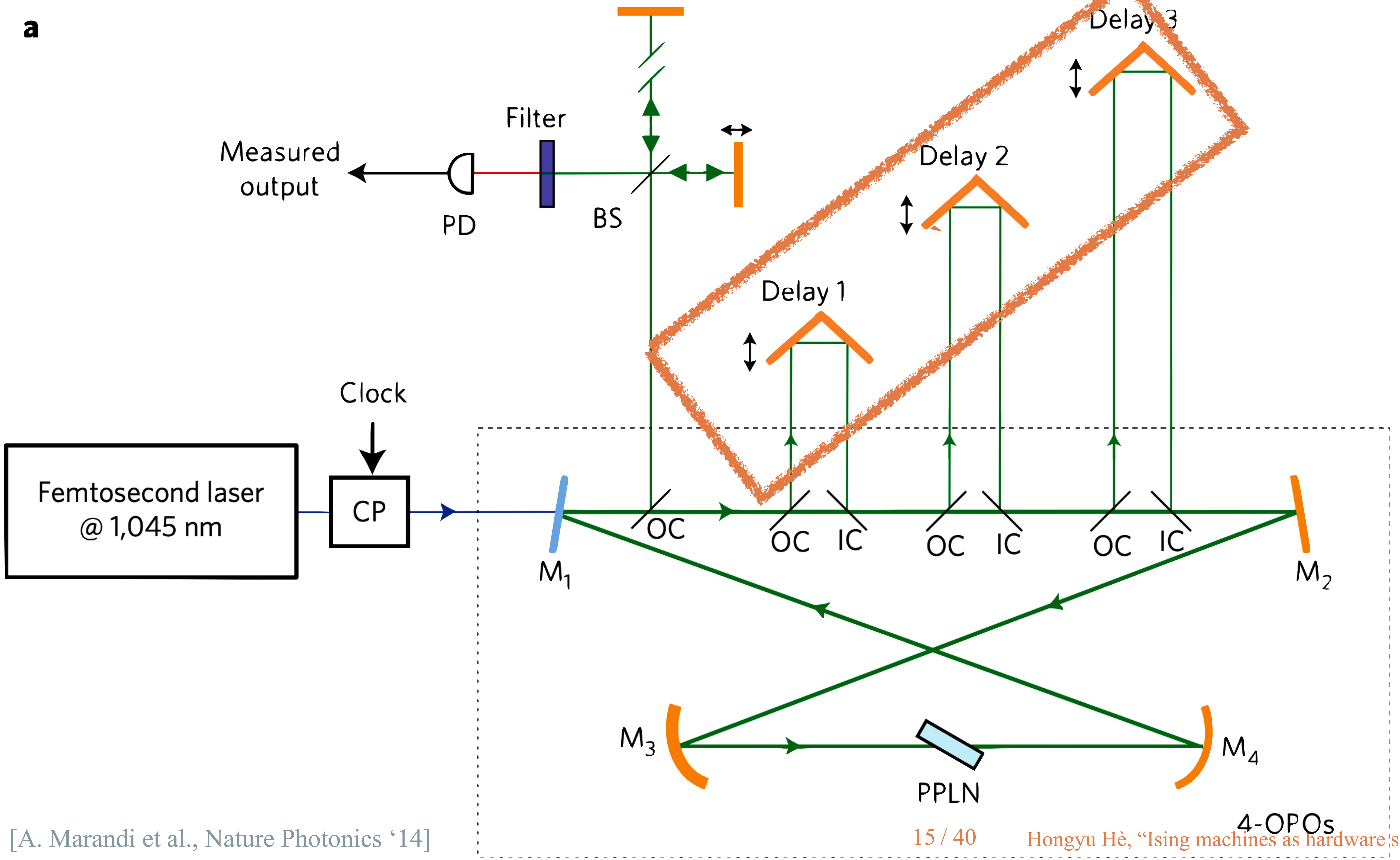


[A. Marandi et al., Nature Photonics ‘14]

Programming the CIM

Coupled four DOPOs

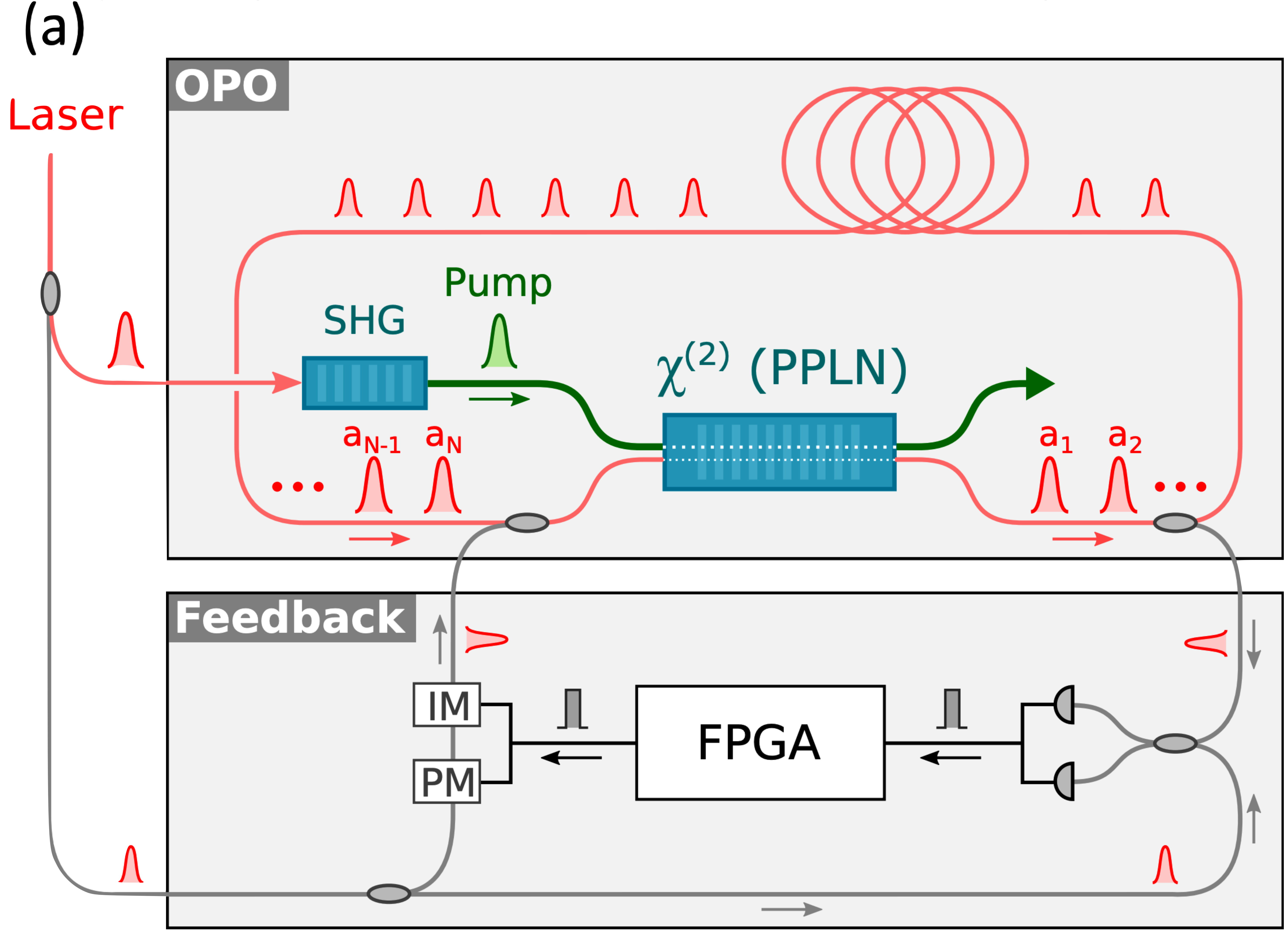




[A. Marandi et al., Nature Photonics '14]

Simulating “Beam Splitters” Using FPGAs

Program arbitrary Ising Hamiltonian without creating optical systems

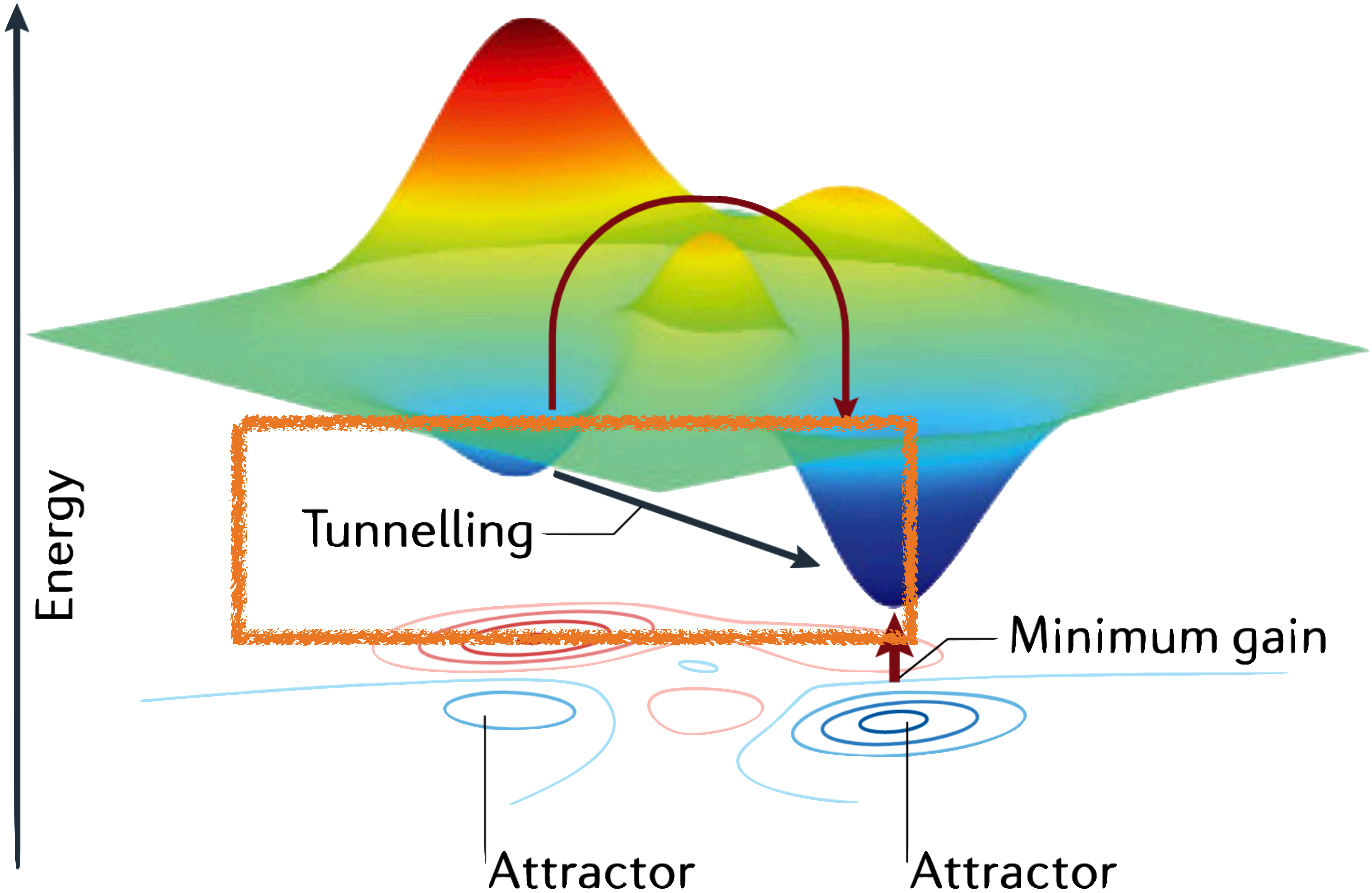


[R Hamerly et al. Science Advances ’19]

Different but still Difficult

CIM does not change the optimization landscape

Nor does it improve its complexity class



Quantum Approaches

Quantum Annealing

$$H(s) = -A(s)\boxed{\sum_{i}^{N} \delta_i^x} + B(s)\boxed{\left[\sum_{i,j}^{N} J_{i,j}\cdot\delta_i^z\delta_j^z + \sum_{i}^{N} h_i\delta_i^z\right]}$$

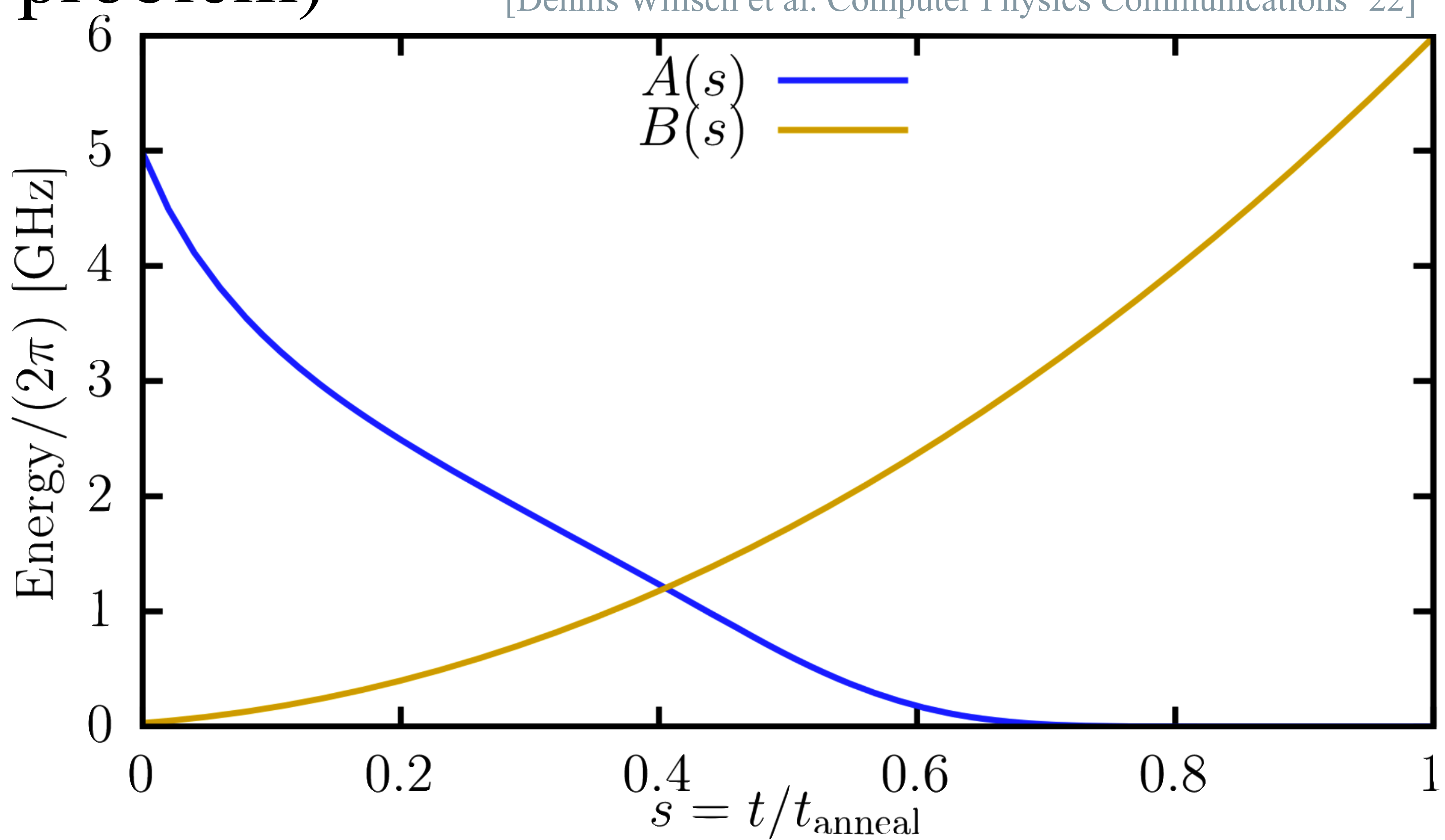
- $\vec{\delta}$: Qubits controlled by quantum operators
- Classical Ising Hamiltonian (the problem)
  - Pauli-Z operator $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: Induces no quantum effect
- **Transverse field**
  - Pauli-X operator: $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: Leads to superposition

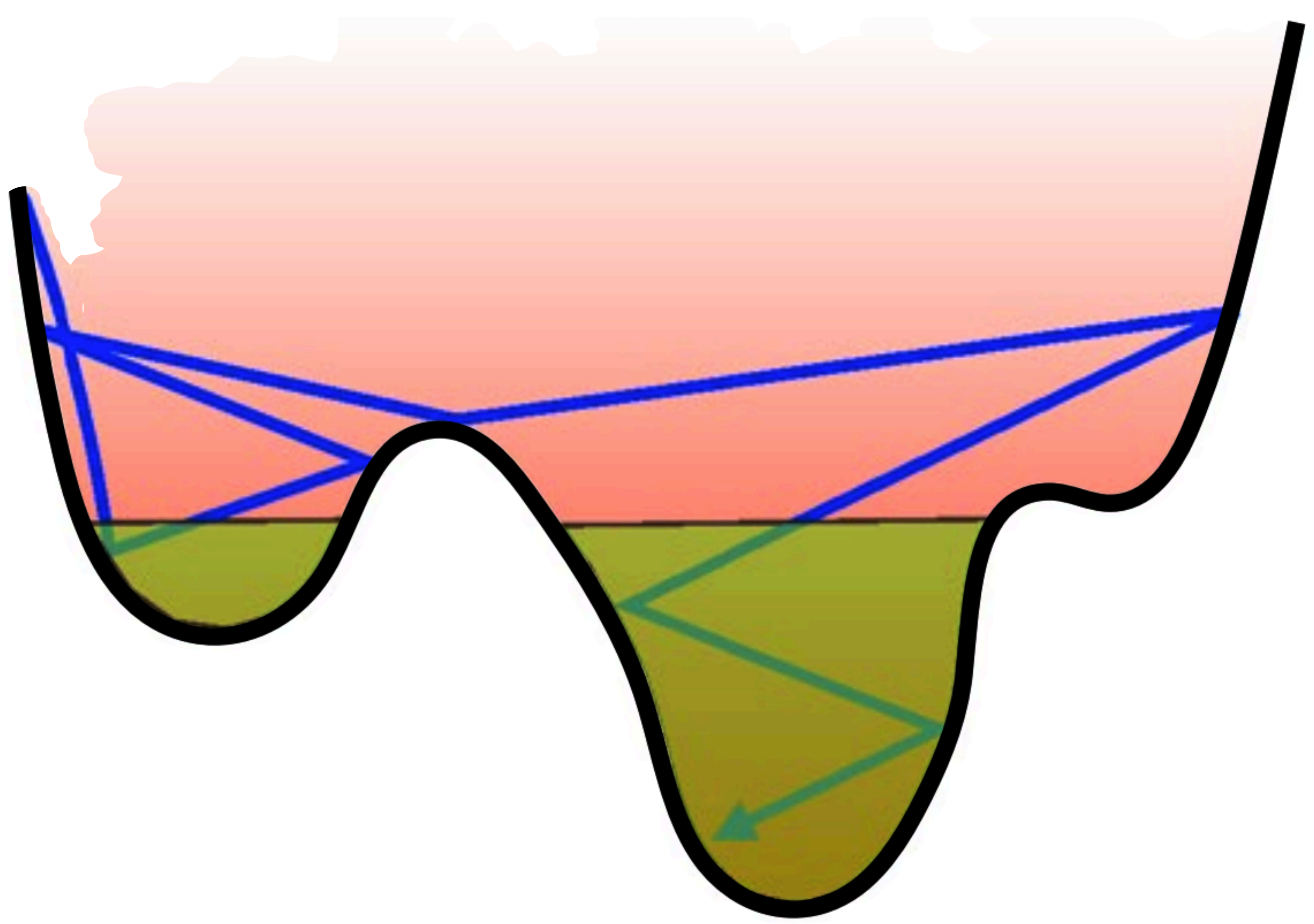
Quantum Annealing

$$H(s) = -A(s)\boxed{\sum_{i}^{N} \delta_i^x} + B(s)\boxed{\left[\sum_{i,j}^{N} J_{i,j}\cdot\delta_i^z\delta_j^z + \sum_{i}^{N} h_i\delta_i^z\right]}$$

- Classical Ising Hamiltonian (the problem)
- Transverse field

[Dennis Willsch et al. Computer Physics Communications ’22]

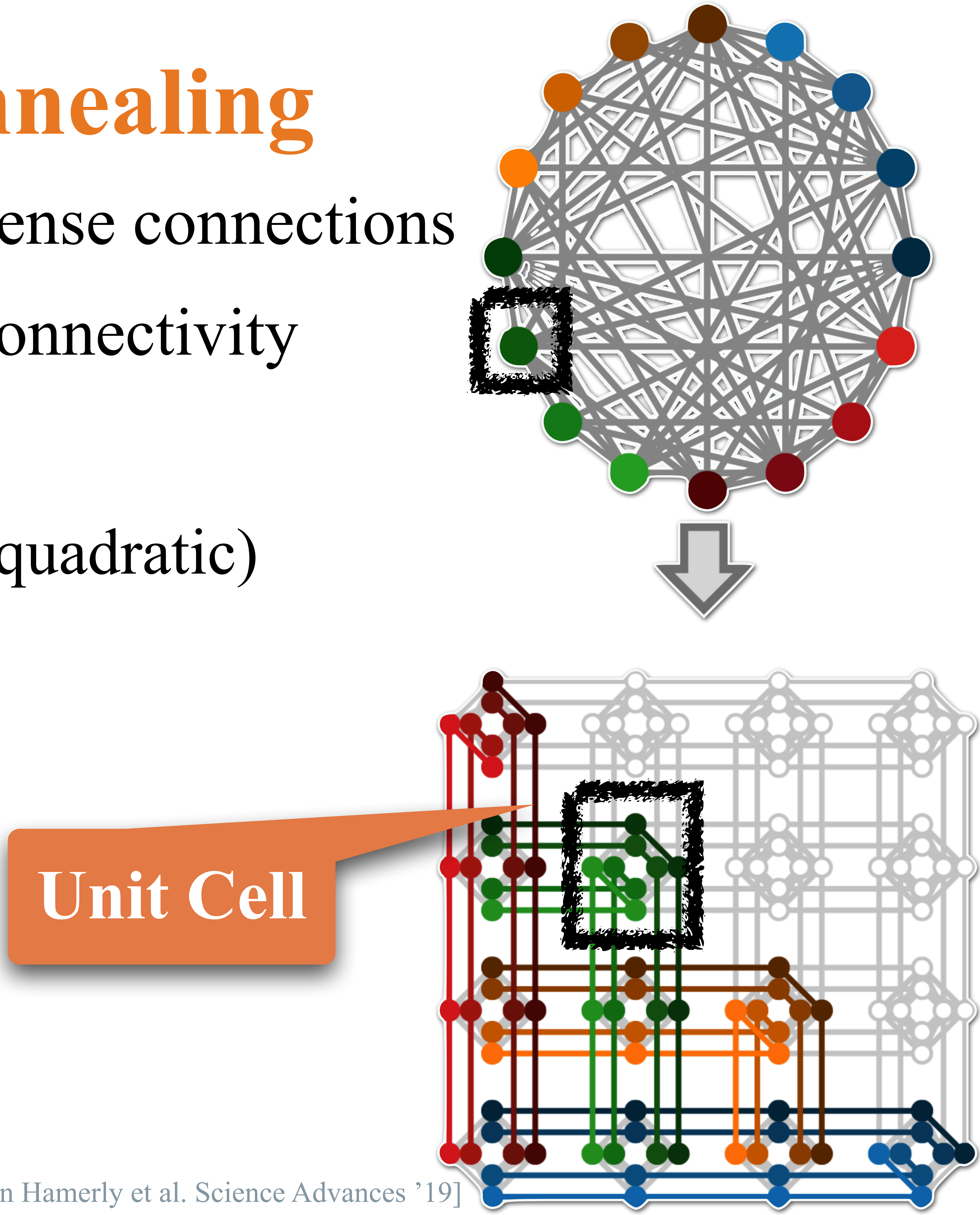




[Roland Sandt et al. Scientific Reports ’23]

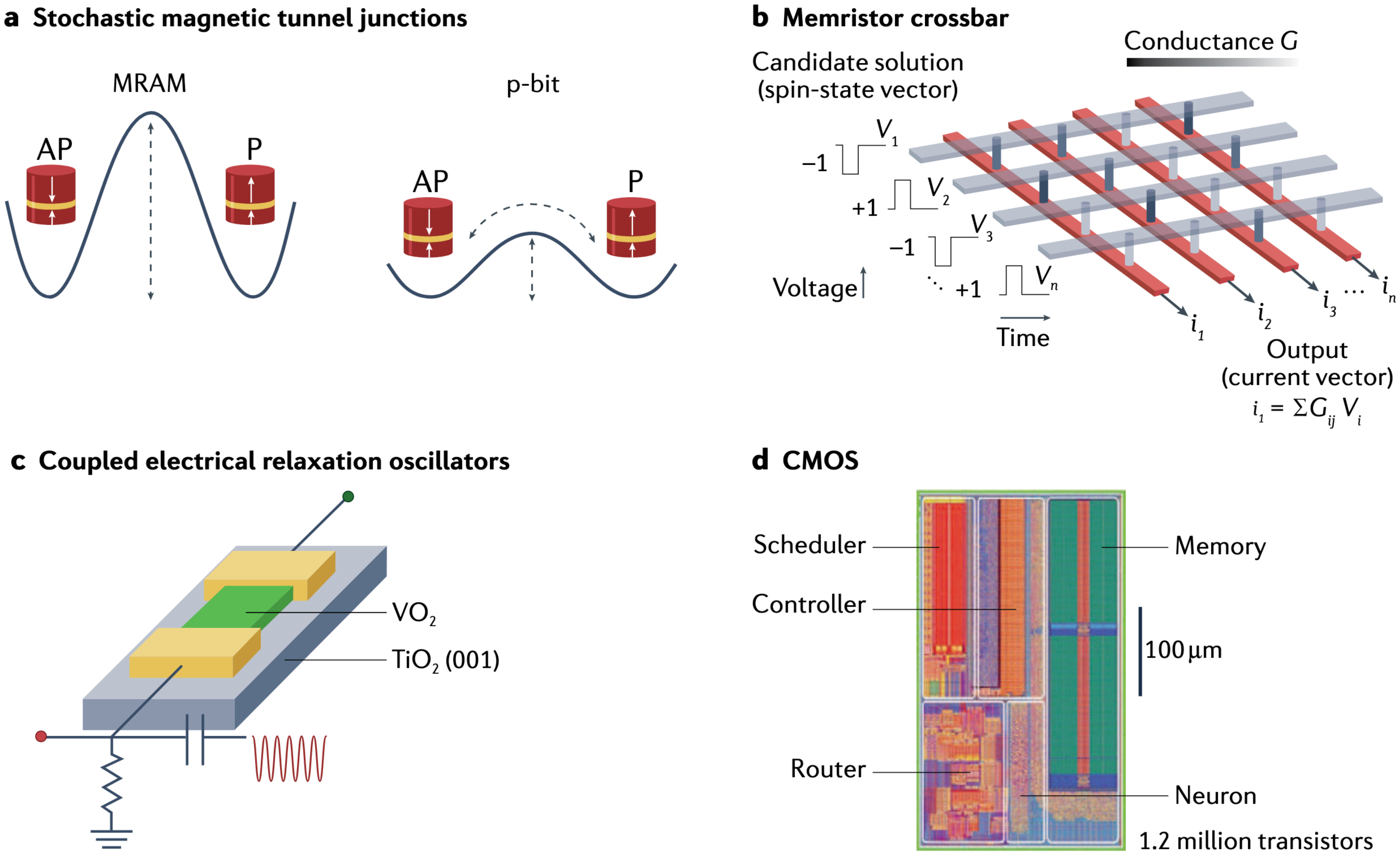
Problems with Quantum Annealing

- Ising problems often require **all-to-all** dense connections
- (D-Wave) Hardware limited to **sparse** connectivity
- One node mapped to a Chimera graph
  - 1 Ising spin : N hardware qubits (often quadratic)
  - 2K qubits ⇒ ≤50 spins
- **Quadratic atop exponential runtime!**



[Ryan Hamerly et al. Science Advances ’19]

Other Approaches



Evaluation & Comparisons

Evaluation Metrics

- Computational complexity ✗
  - Most Ising machines are heuristic solvers (no theoretical guarantees)
  - Still $O(b^{a \cdot n})$ for near-optimal solutions
    - Improve on b or a
- Empirical performance
  - p_{suc}: Probability of finding exact ground state in one shot
    - Not time consideration (longer run, $p_{\text{suc}}\uparrow$)
  - Time-to-solution $T_{\text{sol}} = \tau \dfrac{\ln 0.01}{\ln(1 - p_{\text{suc}})}$

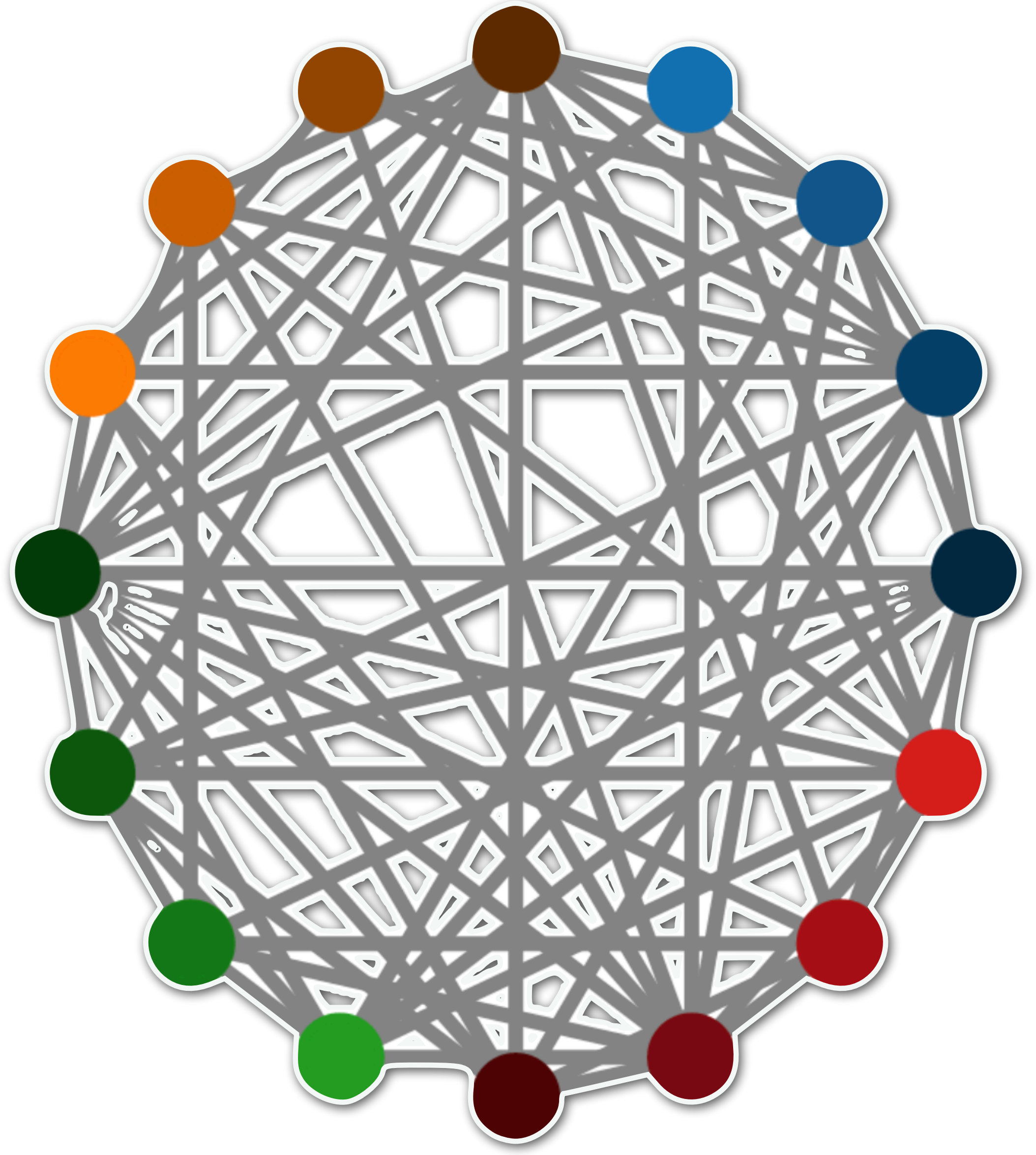
Evaluation Workloads

1) Dense MaxCut instances

2) Sherrington-Kirkpatrick (SK) problems

$$H_{\text{SK}} = -\sum_{i<j} J_{ij} \cdot s_i s_j$$

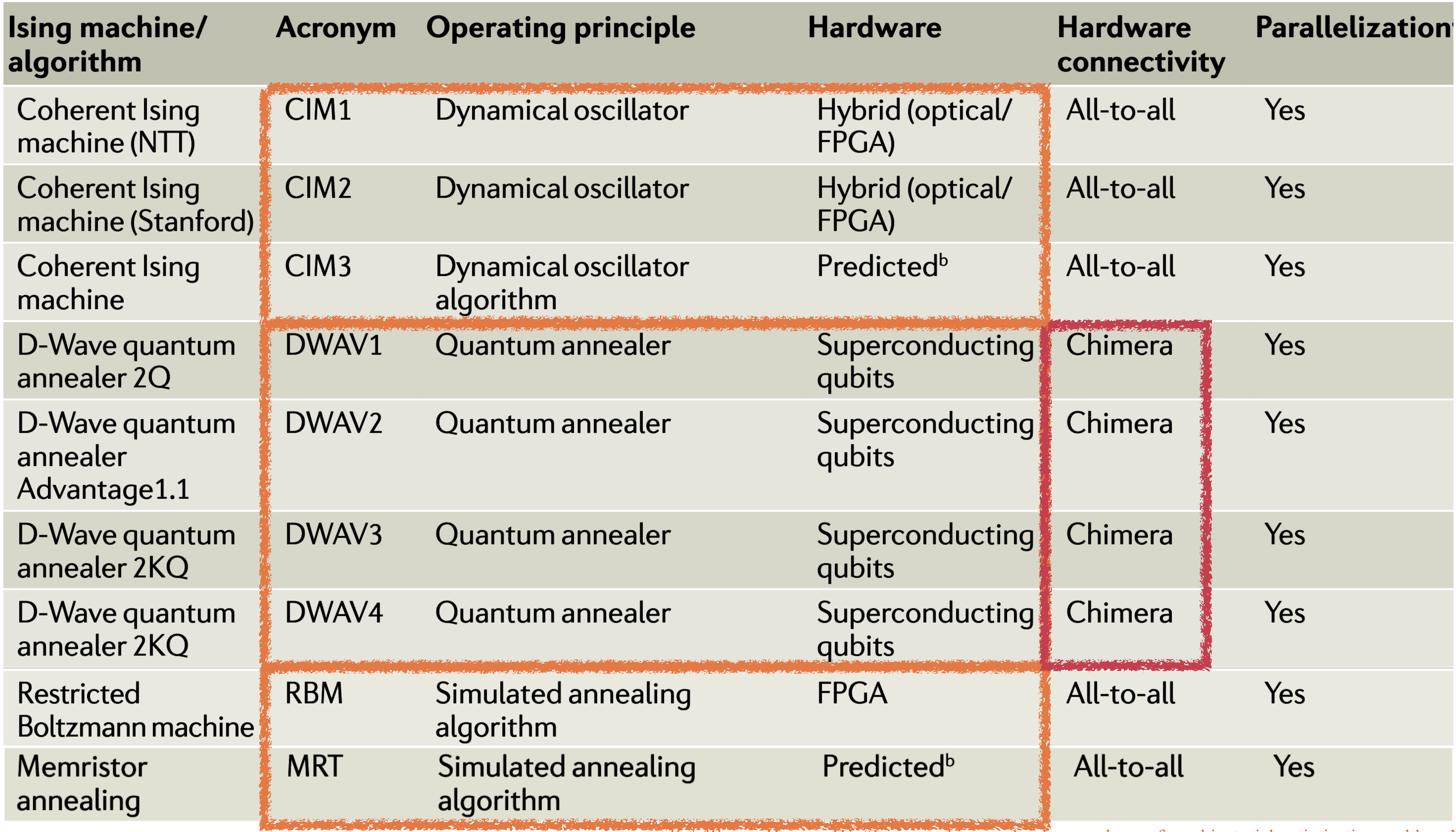
(compared to classical Hamiltonian)



- Fixed positive values in J → Random +1 or −1 couplings
- Local interactions → Full connectivity
- …

Results are **collected** from various original studies

Compared Ising Machines (1)



Ising machine/ algorithm	Acronym	Operating principle	Hardware	Hardware connectivity	Parallelization
Coherent Ising machine (NTT)	CIM1	Dynamical oscillator	Hybrid (optical/ FPGA)	All-to-all	Yes
Coherent Ising machine (Stanford)	CIM2	Dynamical oscillator	Hybrid (optical/ FPGA)	All-to-all	Yes
Coherent Ising machine	CIM3	Dynamical oscillator algorithm	Predicted[b]	All-to-all	Yes
D-Wave quantum annealer 2Q	DWAV1	Quantum annealer	Superconducting qubits	Chimera	Yes
D-Wave quantum annealer Advantage1.1	DWAV2	Quantum annealer	Superconducting qubits	Chimera	Yes
D-Wave quantum annealer 2KQ	DWAV3	Quantum annealer	Superconducting qubits	Chimera	Yes
D-Wave quantum annealer 2KQ	DWAV4	Quantum annealer	Superconducting qubits	Chimera	Yes
Restricted Boltzmann machine	RBM	Simulated annealing algorithm	FPGA	All-to-all	Yes
Memristor annealing	MRT	Simulated annealing algorithm	Predicted[b]	All-to-all	Yes

Hongyu He, “Ising machines as hardware solvers of combinatorial optimization problems” (paper review)

Success Probability

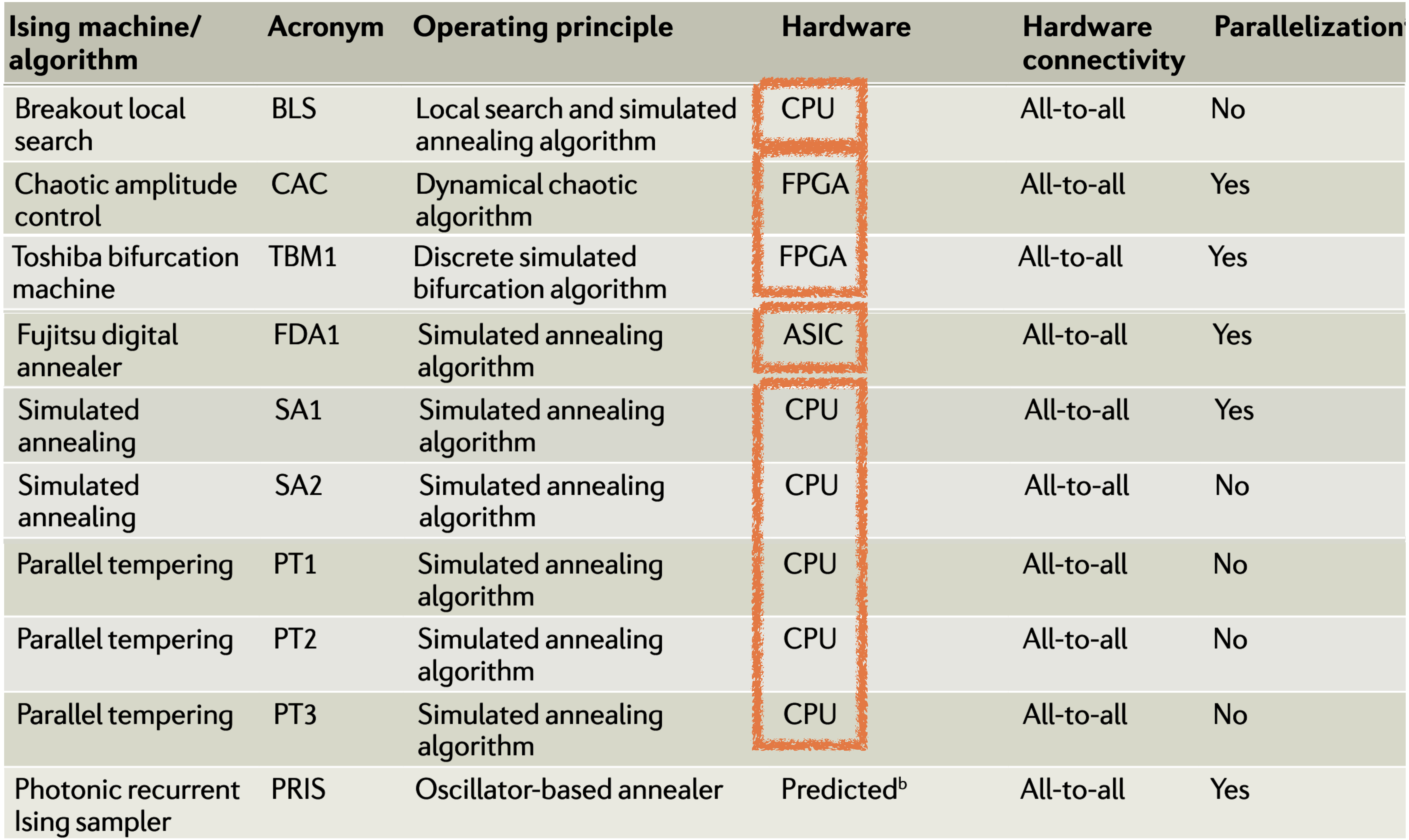
Best scaling with N:
$p_{\mathbf{suc}} \propto \exp\{-bN\}$

Comparable

Worst:
$p_{\mathbf{suc}} \propto \exp\{-bN^2\}$

(Quadratic atop exponential)

Compared Ising Machines (2)



Ising machine/ algorithm	Acronym	Operating principle	Hardware	Hardware connectivity	Parallelization
Breakout local search	BLS	Local search and simulated annealing algorithm	CPU	All-to-all	No
Chaotic amplitude control	CAC	Dynamical chaotic algorithm	FPGA	All-to-all	Yes
Toshiba bifurcation machine	TBM1	Discrete simulated bifurcation algorithm	FPGA	All-to-all	Yes
Fujitsu digital annealer	FDA1	Simulated annealing algorithm	ASIC	All-to-all	Yes
Simulated annealing	SA1	Simulated annealing algorithm	CPU	All-to-all	Yes
Simulated annealing	SA2	Simulated annealing algorithm	CPU	All-to-all	No
Parallel tempering	PT1	Simulated annealing algorithm	CPU	All-to-all	No
Parallel tempering	PT2	Simulated annealing algorithm	CPU	All-to-all	No
Parallel tempering	PT3	Simulated annealing algorithm	CPU	All-to-all	No
Photonic recurrent Ising sampler	PRIS	Oscillator-based annealer	Predicted[b]	All-to-all	Yes

Time To Solution ($p_{\text{suc}} = 0.99$)

$T_{\text{sol}} \propto \exp\{c\sqrt{N}\}$

$T_{\text{sol}} \propto \exp\{cN\}$

Classical digital hardware

Class Discussion

Conclusion

- Connectivity is crucial for Ising machines
- Quantum annealing (QA) is limited by implementation
    - QA computational mechanism works in simulation ✅
    - D-Wave hardware does not scale ❌
    - Quantum mechanics (e.g., entanglement) in QA ❓
- The best: Classical digital methods (‘currently’)
    - Analogue and quantum approaches rapidly developing
    - QA is new; Quantum+Classical ❓

Backup slides …